# The Erdős-Rényi Process Phase Transition Part I: The Coarse Scaling Random Graph Processes Austin

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*Working with Paul Erdős was like taking a walk in the hills. Every time when I thought that we had achieved our goal and deserved a rest, Paul pointed to the top of another hill and off we would go.*
*– Fan Chung*

# The Erdős-Rényi Processes

Begin with empty graph on $n$ vertices.
Each round add one randomly chosen edge
Erdős-Rényi Time: $\frac{n}{2}$ rounds is $t = 1$

PHASE TRANSITION at $t_c = 1$

Modern: $G(n, p)$ with $p = \frac{t}{n}$.

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$t < t_c = 1$
$|C_1| = O(\ln n)$
All $C$ simple [1]

[1] Simple = Tree or Unicylic

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Supercritical

$t > t_c = 1$

GIANT COMPONENT

$|C_1| = \Theta(n)$

Complex (= Not Simple)

All *other* $C$ simple

All *other* $|C| = O(\ln n)$

[1] Simple = Tree or Unicylic

# Phase Transition Near Criticality

`Caution:` Double Limits!

Barely Subcritical

$t = 1 - \epsilon$

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Barely Supercritical
$t > 1 + \epsilon$
GIANT COMPONENT
$|C_1| \sim 2\epsilon n$
Complex ($=$ Not Simple)
All *other* $C$ simple
$|C_2| = O(\epsilon^{-2} \ln n)$

# Galton-Watson Birth Process

Begin with Eve

Eve has Poisson mean $\lambda$ children

All children same. Final tree $T$.

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$\lambda > \lambda_c = 1$
INFINITE TREE
$\Pr[T = \infty] > 1$

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GW $\lambda$ roughly $|C|$ at time $t = \lambda$

# A Useful Non-Rigorous Argument

Erdős-Rényi Process.

When $C, C'$ merge, $S \leftarrow S + \frac{2}{n}|C| \cdot |C'|$

$S(t + \frac{2}{n}) - S(t) = \frac{2}{n} \sum_{C \neq C'} \frac{|C|}{n} \frac{|C'|}{n} |C| \cdot |C'|$

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$S(t) = (1-t)^{-1}$

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$S(t) = (1-t)^{-1}$ Critical $t_c = 1$ when $S(t) \to \infty$

# Fictitious Continutation

$X_1, X_2, \ldots$ mutually independent, $X_i \sim Pois(\lambda)$
$i$-th node has $X_i$ children and dies
$Y_t =$ number of living children, $Y_0 = 1$, $Y_t = Y_{t-1} + X_t - 1$
Example: $2, 1, 0, 1, 0, 2, \ldots$
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Erin has no children ($X_5 = 0, Y_5 = 0$) $T = 5$
Fictitous Continuation (convenient!)
Fiona (no parent) has two children ($X_6 = 2, Y_6 = 1$))
Never Ends. $T = \min t$ with $X_t = 0$ (or $T = \infty$)
History $(X_1, \ldots, X_t)$.

# The Queue

Queue size $Y_0 = 1$; $Y_t = Y_{t-1} + X_t - 1$
Tree size $T = T_\lambda$: minimal $t$, $Y_t = 0$. (Maybe $T = \infty$.)
**Theorem:** (Proof later!)

$$\Pr[T_\lambda = k] = \frac{e^{-\lambda k}(\lambda k)^{k-1}}{k!}$$

Critical: $\Pr[T_1 = k] \sim (2\pi)^{-1/2} k^{-3/2}$.
Heavy Tail. $E[T_1] = \infty$.
Comparing: $\Pr[T_\lambda = k] = \Pr[T_1 = k]\lambda^{-1}(\lambda e^{1-\lambda})^k$
NonCritical: $\lambda e^{1-\lambda} < 1$. Exponential tail.

# Immortality

$x = \Pr[T = \infty]$
The Amazing Property: If $Po(\lambda)$ children, each type $\sigma$ with probability $p_\sigma$ – equivalently $Po(\lambda x_\sigma)$ children of type $\sigma$, independently.

Infinite iff at least one child has infinite tree.

$$x = \Pr[Po(\lambda x) \neq 0] = 1 - e^{-\lambda x}$$

Subcritical. $\lambda < 1$. $x = \Pr[T = \infty] = 0$.
Critical. $\lambda = 1$. $x = \Pr[T = \infty] = 0$, $E[T] = \infty$.
SuperCritical. $\lambda > 1$. $x = \Pr[T = \infty]$ is positive solution to equation.

# Creating a Component

Initial $t = 0$: Queue $Y_0 = 1$; Neutral $N_0 = n - 1$.
BFS finds $X_t$ new vertices and adds them to queue
$X_t = BIN[N_{t-1}, p]$; $Y_t = Y_{t-1} + X_t - 1$; $N_t = N_t - N_{t-1} - X_t$
Fictional Continuation
$T = \min t$ with $Y_t = 0$. (Always $T \leq n$.)
Component $C(v)$ has size $T$.
$N_t \sim BIN[n-1, (1-p)^t]$ (BFS *backwards*)
History $(X_1, \ldots, X_T)$

# Graphs Components & Galton-Watson

$T^{GR}$ := size of $C(v)$ in $G(n, \frac{\lambda}{n})$
$T^{PO}$ := size of tree in Galton-Watson process
`Poisson Property:` For any *constants* $c, k$

$$\lim_{n\to\infty} \Pr[BIN[n-c, \frac{\lambda}{n}] = k] = \Pr[Po(\lambda) = k]$$

**Theorem:** For any possible history $H = (x_1, \ldots, x_t)$ the limit, as $n \to \infty$ of the probability of history $H$ in the graph process is the probability of history $H$ is the Galton-Watson process.
**Corollary:** For any *fixed* $\lambda, k$

$$\lim_{n\to\infty} \Pr[T^{GR} = k] = \Pr[T^{PO} = k]$$

# An Unusual Proof

**Theorem:**

$$\Pr[T_\lambda = k] = \frac{e^{-\lambda k}(\lambda k)^{k-1}}{k!}$$

**Proof:** In $G(n,p)$ with $p = \frac{\lambda}{n}$

$$\Pr[|C(v)| = k] = \binom{n}{k-1}(1-p)^{k(n-k)} \Pr[G(k,p) \text{ connected}]$$

For $k$ fixed, $p \to 0$

$$\Pr[G(k,p) \text{ connected}] \sim k^{k-2} p^{k-1}$$

via Cayley's Theorem.

$$\lim_{n\to\infty} \binom{n}{k-1}(1-p)^{k(n-k)} k^{k-2} p^{k-1} = \frac{e^{-\lambda k}(\lambda k)^{k-1}}{k!}$$

and

$$\Pr[T_\lambda = k] = \lim_{n\to\infty} \Pr[|C(v)| = k]$$

gives the theorem!

# Duality

$d < 1 < c$ dual if $de^{-d} = ce^{-c}$
$T_c^{PO}$ conditioned on being finite is $T_d^{PO}$.
Roughly: $G(n, \frac{c}{n}$ with giant component removed is $G(m, \frac{d}{m})$.

# A Convenience

In the graph process:
To avoid technical calculations we shall replace

$$BIN[N_{t-1}, p]$$

with

$$Po[N_{t-1}p]$$

in finding the number of "new" vertices.

# The Subcritical Regime $\lambda < 1$

$T = |C(v)|$ is stochastically dominated by taking $BIN[n-1, p] \sim Po[\lambda]$ new vertices at each step.
$\Pr[|C(v)| \geq k] \leq \Pr[T_\lambda^{PO} \geq k]$ Exponential decay. For $k = K \ln n$,

$$\Pr[|C(v)| \geq k] = o(n^{-1})$$

so that

$$|C_{MAX}| \leq K \ln n$$

# The Supercritical Regime $\lambda > 1$

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**EXISTENCE**

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**EXISTENCE**

**UNIQUENESS**

# No Middle Ground

Fictional Continuation gives

$$\Pr[|C(v)| = t] \le \Pr[|N_t| = n - t] = \Pr[BIN[n-1,(1-p)^t] = n - t]$$

`Case I:` $t = o(n)$. $1 - (1-p)^t \sim pt = \lambda t/n$.

$$\Pr[BIN[n-1, \lambda t/n] \le t - 1] \text{ drops exponentially in } t$$

For $t \ge K \ln n$, $\Pr[|C(v)| = t] = o(n^{-10})$ `Case II:` $t \sim yn$.
$1 - (1-p)^t \sim 1 - e^{-\lambda y}$.

$$\Pr[BIN[n-1, 1 - e^{-\lambda y}] \sim yn] \text{is exponentially small}$$

*unless* $y = \Pr[T_\lambda^{PO} = \infty]$.
Hence whp for *all* vertices $v$ either

$$(SMALL)|C(v)| \le K \ln n$$

or

$$(GIANT)|C(v)| \sim yn$$

## Sandwiching the Graph Process

$\Pr[|C(v)| \geq t]$ is bounded from *above* by the Galton-Watson with $BIN[n-1,p]$ children.
$\Pr[|C(v)| \geq t]$ is bounded from *below* by the Galton-Watson with $BIN[n-t,p]$ children.
With $t \leq K \ln n$ bounds asymptotic.

$$\Pr[|C(v)| \geq K \ln n] \sim \Pr[T_\lambda^{PO} \geq K \ln n] \sim \Pr[T_\lambda^{PO} = \infty] = y(\lambda)$$

$S$ = number of $v$ with $C(v)$ SMALL. $E[S] \sim (1-y)n$.
Variance calculation: $S \sim (1-y)n$ whp.
If not SMALL then GIANT.
`Conclusion:` There exists a unique GIANT component of size $\sim yn$ and all other components are SMALL.

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Each round randomly chosen edge $\{v, w\}$

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Example of Achlioptas process.

Power of choice.

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- ???

# Susceptibility

$$S(G) = \frac{1}{n}\sum_{C} |C|^2 = E[|C(v)|]$$

Infinite Grid: $\chi = E[|C(\vec{0})|]$

$S(t)$ is $S(G)$ at time $t$.

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Select first edge, $x_1 \leftarrow x_1 - \frac{2}{n}$

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$$x_1(t + \tfrac{2}{n}) - x_1(t) = \; - x_1^2(t)\tfrac{2}{n} - (1 - x_1^2(t))\tfrac{2x_1(t)}{n}$$

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$x_1' = -x_1^2 - (1 - x_1^2)(x_1)$. $x_1(0) = 1$. Smooth function.

# Susceptibility for Bohman-Frieze II

$$S(t + \tfrac{2}{n}) - S(t) =$$

# Susceptibility for Bohman-Frieze II

$$S(t + \tfrac{2}{n}) - S(t) = x_1^2(t)\tfrac{2}{n}$$

Select first edge, $S \leftarrow S + \frac{2}{n}$

# Susceptibility for Bohman-Frieze II

$$S(t+\tfrac{2}{n}) - S(t) = x_1^2(t)\tfrac{2}{n} + (1 - x_1^2(t))\tfrac{2S^2(t)}{n}$$

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Select second random edge, $S \leftarrow S + \frac{2S^2}{n}$

$S' = -x_1^2(1) - (1 - x_1^2(t))S^2$. $S(0) = 1$. Explodes at $t_c \sim 1.1763$

Theorem: (Wormald-JS) Giant Component appears at $t_c$.

Analogue: $p_c$ for $E[|C(\vec{0})|] = \infty$ same as $p_c$ for infinite component.

*Comstock grins and says, "You sound awfully sure of yourself, Waterhouse! I wonder if you can get me to feel that same level of confidence."*
*Waterhouse frowns at the coffee mug. "Well, it's all in the math," he says, " If the math works, why then you* should *be sure of yourself. That's the whole point of math."*
*from* Cryptonomicon *by Neal Stephenson*